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1982 J. Phys. A: Math. Gen. 15 1865

(http://iopscience.iop.org/0305-4470/15/6/027)

J. Phys. A: Math. Gen. 15 (1982) 1865–1872. Printed in Great Britain

Directed percolation: shape of the percolation cone, conductivity exponents, and high-dimensionality behaviour

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Received 8 October 1981, in final form 29 January 1982

Abstract. This paper presents several results for the directed percolation problem. The shape of the percolation cone is discussed, showing how the opening angle behaves near threshold, and explaining the orientation dependence of the correlation exponents found by Domany and Kinzel. The Cayley tree model is used to discuss the high-dimensionality limit. The conductivity problem on a directed network is introduced, and the high-dimensionality values for the conductivity exponents s and τ are found.

1. Introduction

A directed network is a network which has a direction assigned to some or all of its bonds. Directed percolation is a generalisation of the percolation process to a directed network. In this generalisation it is relevant to determine the set of network sites that can be attained from a given starting site, when the directed bonds can only be used in their direction. In this case the relationship of attainment is not commutative: 'A can be attained from B' does not imply the converse.

In the models to be discussed in §§ 2–4, the network is embedded in a lattice, all bonds are parallel to one or another of the principle axes of the lattice, and the bonds are directed in the positive sense of the relevant axis. Then the set of points attainable from a given origin is restricted to a sector of space (e.g. in two directions, this would be a quadrant), and the set of points attainable in exactly t steps lies on a plane through this sector. It is sometimes convenient to discuss this model as decribing the evolution in time t of a system in $D = d-1$ transverse dimensions.

When the bonds are chosen to be present with probability p (and otherwise random) there is a concentration p_c which divides two distinct regimes. (i) For $p<p_c$, all clusters of attainable sites (from any origin) are finite. The clusters are anisotropic: the extent in the t direction is characterised by $\xi_\| \sim (p_c-p)^{-\nu_\|}$, whereas the transverse width is $\xi_\perp \sim (p_c-p)^{-\nu_\perp}$, with $\nu_\| > \nu_\perp$. (ii) For $p>p_c$, there are clusters of infinite extent, which are roughly localised to a conical region. Figure 1 shows an example: the ensemble-averaged probability that a site is attainable in t steps from the origin in $d=2$ directed percolation. The abscissa is the one transverse coordinate. The parameters which characterise this cluster are as follows.

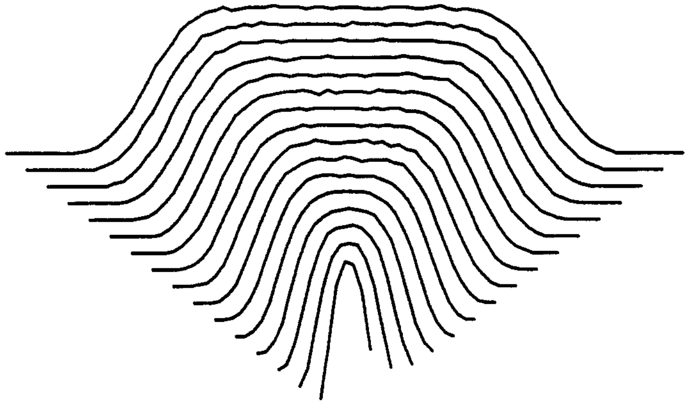

Figure 1. Two-dimensional directed percolation. Here is plotted the probability of attaining a point with transverse coordinate x at 'time' t, for $t=5, 10, \ldots, 80$. The horizontal coordinate is x; the curves have been displaced vertically to eliminate overlap, with the earliest t at the bottom. The curves are all interpreted as a square step of width $2vt$, with edges blurred by an amount w which increases with t.

(1) The two correlation lengths

$$\xi_{\|} \sim (p-p_c)^{-\nu_{\|}} \qquad \xi_{\perp} \sim (p-p_c)^{-\nu_{\perp}} \tag{1.1}$$

determine the scale of inhomogeneity of the clusters (a given cluster will contain holes of width $\xi_{\perp}$ and length $\xi_{\|}$). The effects of these are only indirectly visible in figure 1.

(2) The cone width x is proportional to t

$$x = vt \tag{1.2}$$

where v determines the opening angle of the cone. We shall argue in § 2 that $v \sim \xi_{\perp}/\xi_{\|}$ and thus goes to zero at the percolation threshold.

(3) The width w of the transition region at the edge of the cone will also be discussed in § 2. There it will be argued that this feature is attributable to the random walk nature of percolation paths near threshold, and that $w \sim t^{1/2}$.

The exponents $\nu_{\perp}$, $\nu_{\|}$ which characterise these parameters are dimensionality dependent below the upper critical dimensionality, which is $d^*=5$ ($D^*=4$) for this problem (Obukhov 1980, Straley 1982). Above d^* the exponents are independent of dimensionality, and are conveniently calculated using the Cayley tree model (which is infinite dimensional). This is done in § 3. The results ($\nu_{\perp}^*=\frac{1}{2}$, $\nu_{\|}^*=1$) are in agreement with other estimates for these quantities (Obukhov 1980).

Section 4 defines and discusses the problems of conductivity on a percolating directed lattice. The principal result is the value for the conduction exponents in the high-dimensionality limit: $s^*=1$, $\tau^*=2$. It is interesting and relevant to note that these values satisfy the relation

$$s^*+\tau^*=(d^*-1)\nu_{\perp}+\nu_{\|} \tag{1.3}$$

which is the appropriate generalisation of the conduction exponent hyperscaling relation conjectured (Straley 1980a).

Directed percolation is closely related to Reggeon field theory. The implications of the present work for the latter subject are mentioned in § 5.

2. Shape of the percolating cluster

2.1. *The* SSDG *model*

It is useful to follow Skal and Shklovskii (1974) and de Gennes (1976) in categorising a bond of a network as active if there is a path of infinite extent from the chosen origin through the bond; and to group the active bonds into chains, where a chain is a set of bonds which can be removed from the network by cutting a pair of bonds (cutting just one of these renders the whole chain inactive). The sites where chains join are called nodes. Near the percolation threshold the chains are long and the nodes are spread well apart; specifically, the nodes which terminate a given chain are separated on the average by a distance $\xi_\parallel$ in the t direction and by a distance $\xi_\perp$ in the transverse direction. These lengths diverge at p_c with characteristic exponents $\nu_\parallel$, $\nu_\perp$ (see (1.1)), where $\nu_\parallel > \nu_\perp$, intuitively because the chain performs something like a random walk in the transverse coordinates.

This picture of the structure of the percolating component of the network has direct implications for the cone width x and the width w of the transition region at the edge of the cone (see figure 1). One way to determine the opening angle of the percolation cone would be to construct a sample network and then trace a path through it along the active chains, making the choice at the nodes of the chain that is destined to the largest transverse displacement. In the case $D=1$ ($d=2$) this prescription is simply always to choose the right-hand fork, which will then trace the right-hand extremal edge of the cluster. In general the path is a biased random walk. The cone edge is defined by

$$\tan \phi_c = v = \lim_{t\to\infty} x/t \tag{2.1}$$

According to the SSDG picture, $t \sim n\xi_\parallel$ and $x \sim n\xi_\perp$ (where n is the number of nodes passed), giving $v = \xi_\perp/\xi_\parallel$. The successive x displacements are drawn from a distribution with means $\xi_\perp$ and with uncorrelated fluctuations about this mean also of order $\xi_\perp$. Then $\langle (x-n\xi_\perp)^2\rangle \sim n\xi_\perp^2$, giving a width to the edge of the ensemble-averaged distribution which is proportional to $t^{1/2}$:

$$w \sim \xi_\perp (t/\xi_\parallel)^{1/2}. \tag{2.2}$$

2.2. *Correlations along a ray*

Domany and Kinzel (1981) studied the correlations along infinite strips inclined at some angle ϕ to the t direction for a $D=1$ system. For angles outside the critical cone ($\phi > \phi_c$), the probability $P(R, \phi; p)$ that a site at distance R from the origin is attainable decays to zero; and they defined an orientation-dependent correlation length ξ_ϕ to describe the length scale of this decay. They found $\xi_\phi \sim (p(\phi)-p)^{-2}$, where $p(\phi)$ is the value of p for which the observing angle ϕ is the critical angle ϕ_c.

At first glance this seems a curious result since the percolation exponents ($\nu_\perp = 1.27$, $\nu_\parallel = 1.73$ according to Brower *et al* (1978)) are rather different from 2. However, the result is simply explained as follows.

Consider first the statistics of clusters for given p. The probability that a cluster has its external edge at ϕ is Gaussian distributed about ϕ_c with angular width $\omega = w/t \simeq \xi_\perp \xi_\parallel^{-1/2} t^{-1/2}$, and so the probability that a cluster reaches ϕ (or beyond) is

$$\begin{aligned} P(R, \phi) &\simeq \operatorname{erfc}[(\phi - \phi_c)/\omega] \\ &\simeq \operatorname{erfc}[(\phi - \phi_c) t^{1/2} \xi_\parallel^{1/2} / \xi_\perp] \\ &\simeq \exp[-(\phi - \phi_c)^2 t \xi_\parallel / \xi_\perp^2] \end{aligned} \tag{2.3}$$

for large R, where $t = R \cos \phi$. The scale for t (e.g. the average value of t over this distribution) is of order

$$\xi_\phi \simeq \xi_\perp^2 \xi_\parallel^{-1} (\phi - \phi_c)^{-2}. \tag{2.4}$$

In the calculation of Domany and Kinzel, ϕ is held fixed and ϕ_c is caused to change by varying p. However, for any $\phi \neq 0$, ϕ_c is a non-singular function of p near $p(\phi)$; and $\xi_\perp$ and $\xi_\parallel$ are not singular at $p(\phi)$ ($\neq p_c$), so this result becomes

$$\xi_\phi \simeq \text{constant}(p(\phi) - p)^{-2}. \tag{2.5}$$

2.3. *Numerical simulation*

Figures 1 and 2 illustrate the points made in this section. The data plotted is for the square lattice ($D = 1$); the distribution is for 2000 realisations of clusters for $p = 0.8$, showing the probability that a point is attainable for times $t = 5, 10, 15, \ldots, 80$. Figure 1 plots these data against x to show convergence to a well defined wedge. Figure 2 plots the same data against $(x + vt)/t^{1/2}$, where v was chosen to make the left-hand edges of the distribution overlap; our contention is that their left edges coincide, and that the right-hand edges are all parallel, in agreement with equation (2.2).

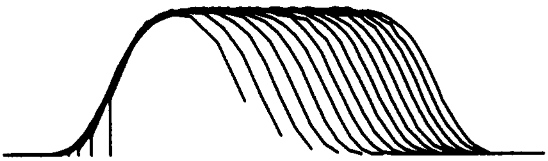

Figure 2. Two-dimensional directed percolation. These are the same data, now plotted against $(x + vt)/t^{1/2}$. It is claimed that the shape of the edges is essentially independent of t.

3. The high-dimensionality limit

The behaviour of a percolating system above the upper critical dimensionality ($D^* = 4$) is conveniently studied using the Cayley tree, which is an infinitely branching network with no closed loops. Here we shall use a directed Cayley tree, embedded in an infinite dimensional space in such a way that every bond has unit component in the t direction, and component $\pm\frac{1}{2}$ in a particular transverse direction (which we shall call x), as well as components in other directions so that different paths which lead to the same coordinates (x, t) always lead to distinct points in space.

Since the path from the origin to a network site is unique, the probability that a site is attainable is just p^t, independent of its transverse coordinates. However, the number of sites having the same x coordinate is $\binom{t}{\frac{1}{2}t+x}$ and therefore the average number of attainable sites with coordinates (x, t) is

$$N(x, t)=\binom{t}{\frac{1}{2}t+x}p^t \approx (2/\pi t)^{1/2}(2p)^t \exp(-2x^2/t). \tag{3.1}$$

3.1. $\xi_\parallel$ and $\xi_\perp$ below p_c

Below p_c all clusters are finite, and have dimensions determined by $\xi_\parallel$ and $\xi_\perp$. From (3.1) we can calculate the number of sites belonging to a cluster,

$$N=\sum N(x, t)=(1-2p)^{-1} \tag{3.2}$$

the average duration of a cluster,

$$\xi_\parallel=N^{-1}\sum tN(x, t)=2p(1-2p)^{-1} \tag{3.3}$$

and the mean-square transverse dimension of a cluster,

$$\xi_\perp^2=N^{-1}\sum x^2N(x, t)=p(1-p)(1-2p)^{-1}. \tag{3.4}$$

All these quantities diverge at $p_c=\frac{1}{2}$, giving

$$\begin{aligned} N &\sim (p_c-p)^{-\gamma} && \text{with} \quad \gamma=1 \\ \xi_\parallel &\sim (p_c-p)^{-\nu_\parallel} && \text{with} \quad \nu_\parallel=1 \\ \xi_\perp &\sim (p_c-p)^{-\nu_\perp} && \text{with} \quad \nu_\perp=\tfrac{1}{2}. \end{aligned} \tag{3.5}$$

The averaging procedure adopted in equations (3.3) and (3.4) strongly favours the larger clusters—in effect a cluster of N sites is counted N times. It might seem more reasonable to average x^2 and t over clusters, counting each cluster only once. However, this latter procedure will give a different answer which cannot be directly identified with the characteristic length scales. This point arose previously in the 'ant in labyrinth' problem and has been discussed by Stauffer (1979) and Straley (1980b).

3.2. v and w above p_c

The geometry of the Cayley tree is sufficiently different from that of a finite dimensionality space that there are some problems in assigning these parameters; as was noted above, the probability that any particular site is occupied is independent of its transverse position. However, the percolation cone comes to exist because there are more paths from the origin to sites along the t direction, and this is the feature which gives the transverse shaping to the distribution (3.1). Thus it is relevant to note that $N(x, t)$ is large for small x, small for large x, and the contour where $N(x, t)=1$ is

$$x^2=\tfrac{1}{2}t^2 \ln 2p+\tfrac{1}{4}t \ln(2/\pi t)\approx t^2(p-p_c) \tag{3.6}$$

giving $v=x/t=(p-p_c)^{1/2}$. Along rays within this cone, $N(rt, t)$ grows exponentially; along rays outside the cone, it decreases. The distribution falls off in the transverse direction with a p-independent scale length $wt^{1/2}$. These results agree with the considerations of § 2.

4. The conductivity problem

Closely associated with a percolation problem is a conductivity problem to determine the average current density in the network in the presence of an applied field E. The conductivity of the network near p_c can be characterised by a power law

$$\sigma \sim (p-p_c)^{\tau} \tag{4.1}$$

The problem can be given a simple discussion by means of the SSDG model. The voltage drop across an active chain will be of order $E\xi_{\|}$, giving

$$J = (E\xi_{\|}/L)\xi_{\perp}^{-d+1} \tag{4.2}$$

where L is the resistance of the chain and $\xi_{\perp}^{d-1}$ is the cross sectional area per active chain. Assigning an exponent $\mathscr{S}$ to L, this gives

$$\tau = (d-1)\nu_{\perp} - \nu_{\|} + \mathscr{S}. \tag{4.3}$$

This result comes to have some predictive value when it is assumed that τ is close to unity for all d. At d^*, this becomes $\tau^* = 2$.

4.1. *The Cayley tree model*

As a check on this last result the Cayley model was considered.

Assume every site of the tree is the intersection of four bonds, two directed inwards and two outwards. The percolation threshold for this network is $p_c = \frac{1}{2}$, since at every site there are only two usable outgoing bonds and on the average at least one of these should be open if percolation is to occur.

We can also readily find the current that flows into the network if one site is held at unit potential and the tree is grounded at infinity. The 'unusable' incoming branches continue to be irrelevant and the network is effectively a tree of coordination number three near its percolation threshold. Then Stinchcombe (1973, 1974) has shown that the probability that an outgoing branch conducts is $4(p-p_c)$, and that if it is conducting, its average conductivity is also of order $p-p_c$.

In the presence of an applied field the problem becomes more complicated. Now most active chains will be carrying current, which implies that some of the 'unusable' branches are actually playing an active role. However, the density of nodes where two active chains merge is the same as the density of sites where an active chain branches in two; the conductivity of a branch will be greater than the estimate of the previous paragraph but only by a factor of 2. The probability that a branch (that is, all the sites that can be attained from a chosen origin starting along a particular bond) is conducting is the same as before.

The conductivity of the network in a field can be estimated by the technique first used for ordinary percolation (Straley 1977). The applied field is represented by sources of EMF inserted in every bond, and a branch of the network is modelled as a conductor σ in parallel with a current source I (which is the current into the branch when the origin is grounded). The joint probability of observing particular values is $P(\sigma, I)$, which will be written as

$$P(\sigma, I) = (1-2\varepsilon)\delta(\sigma)\delta(I) + 2\varepsilon H(\sigma, I) \tag{4.4}$$

where H is a joint distribution for branches that are known to be conducting, and

$\varepsilon = 2(p - p_c)$. A recursion relation for H has the form

$$H(\sigma, I) = 2p(1-2\varepsilon)^2 \int \delta\left(\sigma - \frac{\sigma_1}{1+\sigma_1}\right)\delta\left(I - \frac{E\sigma_1 + I_1}{1+\sigma_1}\right)H(\sigma_1, I_1)$$

$$+2\varepsilon p^2(1-2\varepsilon)^2 \int F(\sigma_1, \sigma_2, I_1, I_2; \sigma, I)H(\sigma_1, I_1)H(\sigma_2 I_2) + O(\varepsilon^2) \quad (4.5)$$

where F describes how currents and conductivities are combined at a node. The average current into a grounded branch is then given by

$$\langle I\rangle_H = \iint IH(\sigma, I) = 2p(1-2\varepsilon)^2\left\langle\frac{E\sigma + I}{\sigma+1}\right\rangle_H + O(\varepsilon). \quad (4.6)$$

Since σ is of order ε, this is

$$\langle(\sigma + 3\varepsilon)I\rangle_H = E\langle\sigma\rangle_H + O(\varepsilon). \quad (4.7)$$

The important point is that $\langle I\rangle$ must be of order $E \times$ unity, since its multiplier and the other side of the equation are both of order ε.

The leading order contribution to the current density in the network is

$$J \approx (2p)^2 \int \frac{\sigma_1 I_2 + \sigma_2 I_1 + \sigma_1\sigma_2 E}{\sigma_1 + \sigma_2 + \sigma_1\sigma_2}(2\varepsilon)^2 H(\sigma_1 I_1)H(\sigma_2 I_2) \sim \varepsilon^2 E \quad (4.8)$$

giving $\tau^* = 2$.

4.2. *The s exponent*

There is a second conductivity problem in which all bonds of the network conduct in the forward direction, but a fraction p of them conduct infinitely well. Now the conductivity will diverge as p_c is approached from below

$$\sigma \sim (p - p_c)^{-s}. \quad (4.9)$$

The current distribution in a Cayley tree near p_c can be estimated by the following argument. The infinite conductivity links will form clusters which are finite but large; the average diameter is $\xi_\| \sim (p - p_c)^{-1}$. The finite conductivity links will be touching two such clusters with probability $1 - (1-p)^2 \sim 75\%$, and so it is clear that the boundary layer between clusters is nowhere much more than a single link wide. Then the boundary layer conductors have a voltage drop of order $\xi_\|$ across them, giving an average current in any bond of the network of the same size. Thus the high-dimensionality limit of s is $s^* = 1$.

5. Connections with Reggeon field theory

The directed percolation problem is closely related to Reggeon field theory (RFT), which is a phenomenological description of the interactions of hadrons (Moshe 1978). Specifically the two theories belong to the same universality class so that there is a one-to-one correspondence between the quantities of the two theories (Grassberger and Sundermeyer 1978, Grassberger and de la Torre 1979, Cardy and Sugar 1980). The transcription is not entirely straightforward: the time-like variable t becomes the rapidity (the logarithm of s, the centre-of-mass energy squared), and the transverse coordinate x is the impact parameter. The threshold for directed percolation corresponds to the case that the cubic interaction is sufficiently large compared to the bare Reggeon gap that the vacuum becomes degenerate; in general the bond probability p is a function of the ratio of the interaction strength to the bare mass. The probability of attaining (x, t) becomes the Green function for scattering at impact parameter x and rapidity $\log s_2 - \log s_1$. The cross sectional area of the percolation cone gives the scattering cross section.

There is also a difference in notation: the natural variables and exponents for the RFT are Δ, α', ν, and z, which are related to the variables discussed in this paper as follows:

$$\Delta = \xi_{\|} \qquad \nu = \nu_{\|} \qquad \alpha' = \xi_{\perp}^2/\xi_{\|} \qquad z = 2\nu_{\|}/\nu_{\perp}.$$

The results of the present paper have the following significance for RFT.

(i) The discussion of the shape of the percolation cone is a discussion of the differential cross section for impact parameters near the edge of the scattering disc. Our result, that this edge is determined by a random walk and thus has essentially an error-function profile, has been previously suggested by Grassberger and de la Torre (1979).

(ii) The Cayley tree model corresponds to an eikonal approximation. This approximation becomes accurate above $D = 4$ because the vertex corrections considered by Abarbanel and Bronzan (1974) are no longer divergent; the loops are then trivially renormalised away.

The conductivity problem discussed in § 4 has no corresponding concept in RFT.

Acknowledgment

This work was supported by the NSF through grant DMR 81-03099.

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